

AN OKA PRINCIPLE FOR STEIN  $G$ -MANIFOLDS

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ABSTRACT. Let  $G$  be a reductive complex Lie group acting holomorphically on Stein manifolds  $X$  and  $Y$ . Let  $p_X: X \rightarrow Q_X$  and  $p_Y: Y \rightarrow Q_Y$  be the quotient mappings. Assume that we have a biholomorphism  $Q := Q_X \rightarrow Q_Y$  and an open cover  $\{U_i\}$  of  $Q$  and  $G$ -biholomorphisms  $\Phi_i: p_X^{-1}(U_i) \rightarrow p_Y^{-1}(U_i)$  inducing the identity on  $U_i$ . There is a sheaf of groups  $\mathcal{A}$  on  $Q$  such that the isomorphism classes of all possible  $Y$  is the cohomology set  $H^1(Q, \mathcal{A})$ . The main question we address is to what extent  $H^1(Q, \mathcal{A})$  contains only topological information. For example, if  $G$  acts freely on  $X$  and  $Y$ , then  $X$  and  $Y$  are principal  $G$ -bundles over  $Q$ , and Grauert's Oka principle says that the set of isomorphism classes of holomorphic principal  $G$ -bundles over  $Q$  is canonically the same as the set of isomorphism classes of topological principal  $G$ -bundles over  $Q$ . We investigate to what extent we have an Oka principle for  $H^1(Q, \mathcal{A})$ .

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## 1. INTRODUCTION

Let  $X$  be a Stein  $G$ -manifold where  $G$  is a complex reductive group. There is a quotient space  $Q_X = X//G$  (or just  $Q$  if  $X$  is understood) and surjective morphism  $p_X$  (or just  $p$ ) from  $X$  to  $Q$ . Then  $Q$  is a reduced normal Stein space and the fibers of  $p$  are canonically affine  $G$ -varieties (generally, neither reduced nor irreducible) containing precisely one closed  $G$ -orbit. For  $S$  a subset of  $Q$  we denote  $p^{-1}(S)$  by  $X_S$  and we abbreviate  $X_{\{q\}}$  as  $X_q$ ,  $q \in Q$ . We have a sheaf of groups  $\mathcal{A}^X$  (or just  $\mathcal{A}$ ) on  $Q$  where  $\mathcal{A}(U) = \text{Aut}_U(X_U)^G$  is the group of holomorphic  $G$ -automorphisms of  $X_U$  which induce the identity map  $\text{Id}_U$  on  $X_U//G = U$ .

Let  $Y$  be another Stein  $G$ -manifold. In [KLS15, KLS] we determined sufficient conditions for  $X$  and  $Y$  to be equivariantly  $G$ -biholomorphic. Clearly we need that  $Q_Y$  is biholomorphic to  $Q_X$ , so let us assume that we have fixed an isomorphism of  $Q_Y$  with  $Q = Q_X$ . Let us also suppose that there are no local obstructions to a  $G$ -biholomorphism of  $X$  and  $Y$  covering  $\text{Id}_Q$ . (See [KLS, Theorem 1.3] for sufficient conditions for vanishing of the local obstructions.) Then there is an open cover  $U_i$  of  $Q$  and  $G$ -biholomorphisms  $\Phi_i: X_{U_i} \rightarrow Y_{U_i}$  inducing  $\text{Id}_{U_i}$ . We say that  $X$  and  $Y$  are *locally  $G$ -biholomorphic over  $Q$* . Set  $\Phi_{ij} = \Phi_i^{-1}\Phi_j$ . Then the  $\Phi_{ij} \in \mathcal{A}(U_i \cap U_j)$  are a 1-cocycle, i.e., an element of  $Z^1(Q, \mathcal{A})$  (we repress explicit mention of the open cover). Conversely, given  $\Psi_{ij} \in Z^1(Q, \mathcal{A})$  (for the same open cover) we can construct a corresponding complex  $G$ -manifold  $Y$  from the disjoint union of the  $X_{U_i}$  by identifying  $X_{U_j}$  and  $X_{U_i}$  over

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$U_i \cap U_j$  via  $\Psi_{ij}$ . By [KLS, Theorem 5.11] the manifold  $Y$  is Stein, and it is obviously locally  $G$ -biholomorphic to  $X$  over  $Q$ . Let  $\Psi'_{ij}$  be another cocycle for  $\{U_i\}$  corresponding to the Stein  $G$ -manifold  $Y'$ . If  $Y'$  is  $G$ -biholomorphic to  $Y$  (inducing  $\text{Id}_Q$ ), then  $\Psi_{ij}$  and  $\Psi'_{ij}$  give the same class in  $H^1(Q, \mathcal{A})$ . Thus  $H^1(Q, \mathcal{A})$  is the set of  $G$ -isomorphism classes of Stein  $G$ -manifolds  $Y$  which are locally  $G$ -biholomorphic to  $X$  over  $Q$  where the  $G$ -isomorphisms are required to induce the identity on  $Q$ .

A fundamental question is whether or not  $H^1(Q, \mathcal{A})$  contains more than topological information. For example, suppose that  $G$  acts freely on  $X$  so that  $X \rightarrow Q$  is a principal  $G$ -bundle. Then  $X$  corresponds to an element of  $H^1(Q, \mathcal{E})$  where  $\mathcal{E}$  is the sheaf of germs of holomorphic mappings of  $Q$  to  $G$ . By Grauert's famous Oka principle [Gra58],  $H^1(Q, \mathcal{E}) \simeq H^1(Q, \mathcal{E}^c)$  where  $\mathcal{E}^c$  is the sheaf of germs of continuous mappings of  $Q$  to  $G$ . In other words, the set of isomorphism classes of holomorphic principal  $G$ -bundles over  $Q$  is the same as the set of isomorphism classes of topological principal  $G$ -bundles over  $Q$ . The main point of this note is to establish a similar Oka principle in our setting.

We define another sheaf of groups  $\mathcal{A}_c$  on  $Q$ . For  $U$  open in  $Q$ ,  $\mathcal{A}_c(U)$  consists of “strongly continuous” families  $\sigma = \{\sigma_q\}$  of  $G$ -automorphisms of the affine  $G$ -varieties  $X_q$ ,  $q \in U$ . We define the notion of strongly continuous family in §3. The sheaf  $\mathcal{A}$  is a subsheaf of  $\mathcal{A}_c$ .

Fix an open cover  $\{U_i\}$  of  $Q$ . Our main theorems are the following (the first of which is a consequence of [KLS, Theorem 1.4]).

**Theorem 1.1.** *Let  $\Phi_{ij}, \Psi_{ij} \in Z^1(Q, \mathcal{A})$  and suppose that there are  $c_i \in \mathcal{A}_c(U_i)$  satisfying  $\Phi_{ij} = c_i \Psi_{ij} c_j^{-1}$ . Then there are  $c'_i \in \mathcal{A}(U_i)$  satisfying the same equation.*

**Theorem 1.2.** *Let  $\Phi_{ij} \in Z^1(Q, \mathcal{A}_c)$ . Then there are  $c_i \in \mathcal{A}_c(U_i)$  such that  $c_i \Phi_{ij} c_j^{-1} \in Z^1(Q, \mathcal{A})$ .*

As a consequence we have the following Oka principle:

**Corollary 1.3.** *The canonical map  $H^1(Q, \mathcal{A}) \rightarrow H^1(Q, \mathcal{A}_c)$  is a bijection.*

*Remark 1.4.* Suppose that  $X$  is a smooth affine  $G$ -variety and that  $Z \rightarrow Q$  is a morphism of affine varieties. Then  $G$  acts on the fiber product  $Z \times_Q X$  and we have the group  $\text{Aut}_{Z, \text{alg}}(Z \times_Q X)^G$  of algebraic  $G$ -automorphisms of  $Z \times_Q X$  which induce the identity on the quotient  $Z$ . A scheme  $\mathcal{G}$  with projection  $\pi: \mathcal{G} \rightarrow Q$  such that the fibers of  $\mathcal{G}$  are groups whose structure depends algebraically on  $q \in Q$  is called a *group scheme over  $Q$* . (See [KS92, Ch. III] for a more precise definition.) We say that *the automorphism group scheme of  $X$  exists* if there is a group scheme  $\mathcal{G}$  over  $Q$  together with a canonical isomorphism of  $\Gamma(Z, Z \times_Q \mathcal{G})$  and  $\text{Aut}_{Z, \text{alg}}(Z \times_Q X)^G$  for all  $Z \rightarrow Q$ . The automorphism group scheme of  $X$  exists (and is an affine variety) if, for example,  $p: X \rightarrow Q$  is flat [KS92, Ch. III Proposition 2.2]. Assuming  $\mathcal{G}$  exists, now consider  $X$  as a Stein  $G$ -manifold and  $\mathcal{G}$  as an analytic variety. Then for  $U$  open in  $Q$ ,  $\mathcal{A}(U) \simeq \Gamma(U, \mathcal{G})$  and one can show that  $\mathcal{A}_c(U)$  is the set of continuous sections of  $\mathcal{G}$  over  $U$ . Thus, in this case, our theorems reduce to the precise analogues of Grauert's for the cohomology of  $\mathcal{G}$  using holomorphic or continuous sections.

For  $U$  an open subset of  $Q$  we have a topology on  $\mathcal{A}_c(U)$  and  $\mathcal{A}(U)$  and we define the notion of a continuous path (or homotopy) in  $\mathcal{A}_c(U)$  or  $\mathcal{A}(U)$ . We establish a result which is well-known in the case of principal bundles but rather non-trivial in our situation.

**Theorem 1.5.** *Let  $\Phi_{ij}(t)$  be a homotopy of elements in  $Z^1(Q, \mathcal{A}_c)$ ,  $t \in [0, 1]$ . Then there are homotopies  $c_i(t) \in \mathcal{A}_c(U_i)$ ,  $t \in [0, 1]$ , such that  $\Phi_{ij}(t) = c_i(t) \Phi_{ij}(0) c_j(t)^{-1}$ . Hence  $\Phi_{ij}(t) \in H^1(Q, \mathcal{A}_c)$  is independent of  $t$ .*

**Theorem 1.6.** *Let  $\Phi_{ij}(t) \in Z^1(Q, \mathcal{A}_c)$  be a homotopy,  $t \in [0, 1]$ , where the  $\Phi_{ij}(0)$  and  $\Phi_{ij}(1)$  are holomorphic. Then there is a homotopy  $\Psi_{ij}(t) \in Z^1(Q, \mathcal{A})$  with  $\Psi_{ij}(0) = \Phi_{ij}(0)$  and  $\Psi_{ij}(1) = \Phi_{ij}(1)$ .*

Here is an outline of this paper. In §2 we recall Luna’s slice theorem and related results. In §3 we define the sheaf of groups  $\mathcal{A}_c$  as well as a corresponding sheaf of Lie algebras  $\mathcal{LA}_c$ . In §4 we show that sections of  $\mathcal{A}_c$  sufficiently close to the identity are the exponentials of sections of  $\mathcal{LA}_c$ . In §5 we establish our main technical result (Theorem 5.1) about homotopies in  $\mathcal{A}_c$ . We prove Theorem 1.1 and Theorem 1.6 as well as a preliminary version of Theorem 1.5. In §6 we establish Theorem 1.2 and use it to prove Theorem 1.5. Finally, let  $X$  and  $Y$  be locally  $G$ -biholomorphic over  $Q$ . We establish a theorem giving necessary and sufficient conditions for a  $G$ -biholomorphism from  $X_U \rightarrow Y_U$  over  $\text{Id}_U$ , where  $U \subset Q$  is Runge, to be the limit of the restrictions to  $X_U$  of  $G$ -biholomorphisms from  $X$  to  $Y$  over  $\text{Id}_Q$ .

*Remark 1.7.* In [KLS15, KLS] we also consider  $G$ -diffeomorphisms  $\Phi$  of  $X$  which induce the identity over  $Q$  and are *strict*. This means that the restriction of  $\Phi$  to  $X_q$ ,  $q \in Q$ , induces an algebraic  $G$ -automorphism of  $(X_q)_{\text{red}}$  where “red” denotes reduced structure. One can adapt the techniques developed here to prove the analogues of our main theorems for strong  $G$ -homeomorphisms replaced by strict  $G$ -diffeomorphisms.

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## 2. BACKGROUND

For details of what follows see [Lun73] and [Sno82, Section 6]. Let  $X$  be a Stein manifold with a holomorphic action of a reductive complex Lie group  $G$ . The categorical quotient  $Q_X = X//G$  of  $X$  by the action of  $G$  is the set of closed orbits in  $X$  with a reduced Stein structure that makes the quotient map  $p_X: X \rightarrow Q_X$  the universal  $G$ -invariant holomorphic map from  $X$  to a Stein space. The quotient  $Q_X$  is normal. When  $X$  is understood, we drop the subscript  $X$  in  $p_X$  and  $Q_X$ . If  $U$  is an open subset of  $Q$ , then  $p^*$  induces isomorphisms of  $\mathbb{C}$ -algebras  $\mathcal{O}_X(X_U)^G \simeq \mathcal{O}_Q(U)$  and  $C^0(X_U)^G \simeq C^0(U)$ . We say that a subset of  $X$  is  $G$ -saturated if it is a union of fibers of  $p$ . If  $X$  is an affine  $G$ -variety, then  $Q$  is just the complex space corresponding to the affine algebraic variety with coordinate ring  $\mathcal{O}_{\text{alg}}(X)^G$ .

Let  $H$  be a reductive subgroup of  $G$  and let  $B$  be an  $H$ -saturated neighborhood of the origin of an  $H$ -module  $W$ . We always assume that  $B$  is Stein, in which case  $B//H$  is also Stein. Let  $G \times^H B$  (or  $T_B$ ) denote the quotient of  $G \times B$  by the (free)  $H$ -action sending  $(g, w)$  to  $(gh^{-1}, hw)$  for  $h \in H$ ,  $g \in G$  and  $w \in B$ . We denote the image of  $(g, w)$  in  $G \times^H B$  by  $[g, w]$ .

Let  $Gx$  be a closed orbit in  $X$ . Then the isotropy group  $G_x$  is reductive and the *slice representation at  $x$*  is the action of  $H = G_x$  on  $W = T_x X / T_x(Gx)$ . By the slice theorem, there is a  $G$ -saturated neighborhood of  $Gx$  which is  $G$ -biholomorphic to  $T_B$  where  $B$  is an  $H$ -saturated neighborhood of  $0 \in W$ .

## 3. STRONGLY CONTINUOUS HOMEOMORPHISMS AND VECTOR FIELDS

The group  $G$  acts on  $\mathcal{O}(X)$ ,  $f \mapsto g \cdot f$ , where  $(g \cdot f)(x) = f(g^{-1}x)$ ,  $x \in X$ ,  $g \in G$ ,  $f \in \mathcal{O}(X)$ . Let  $\mathcal{O}_{\text{fin}}(X)$  denote the set of holomorphic functions  $f$  such that the span of  $\{g \cdot f \mid g \in G\}$  is finite dimensional. They are called the  *$G$ -finite holomorphic functions on  $X$*  and obviously form an  $\mathcal{O}(Q) = \mathcal{O}(X)^G$ -algebra. If  $X$  is a smooth affine  $G$ -variety, then the techniques of [Sch80, Proposition 6.8, Corollary 6.9] show that for  $U \subset Q$  open and Stein we have

$$\mathcal{O}_{\text{fin}}(X_U) \simeq \mathcal{O}(U) \otimes_{\mathcal{O}_{\text{alg}}(Q)} \mathcal{O}_{\text{alg}}(X).$$

Let  $V$  be the direct sum of pairwise non-isomorphic non-trivial  $G$ -modules  $V_1, \dots, V_r$ . Let  $\mathcal{O}(X)_V$  denote the elements of  $\mathcal{O}_{\text{fin}}(X)$  contained in a copy of  $V$ . If  $H$  is a reductive subgroup of  $G$  and  $W$  an  $H$ -module, we similarly define  $\mathcal{O}_{\text{alg}}(T_W)_V$ . Then for  $B$  an  $H$ -saturated neighborhood of  $0 \in W$ ,  $\mathcal{O}_{\text{alg}}(T_W)_V$  generates  $\mathcal{O}(T_B)_V$  over  $\mathcal{O}(B)^H$ . By Nakayama’s Lemma,

$f_1, \dots, f_m \in \mathcal{O}(X)_V$  restrict to minimal generators of the  $\mathcal{O}(U)$ -module  $\mathcal{O}(X_U)_V$  for some neighborhood  $U$  of  $q \in Q$  if and only if the restrictions of the  $f_i$  to  $X_q$  form a basis of  $\mathcal{O}(X_q)_V = \mathcal{O}_{\text{alg}}(X_q)_V$ . Thus by the slice theorem, the sheaf of algebras of  $G$ -finite holomorphic functions is locally finitely generated as an algebra over  $\mathcal{O}_Q$ .

**Definition 3.1.** Let  $U \subset Q$  be relatively compact. Then there is a  $V$  as above such that the  $\mathcal{O}(X_U)_{V_j}$  are finitely generated over  $\mathcal{O}(U)$  and generate  $\mathcal{O}_{\text{fin}}(X_U)$  as  $\mathcal{O}(U)$ -algebra. Let  $f_1, \dots, f_n$  be a generating set of  $\oplus \mathcal{O}(X_U)_{V_j}$  with each  $f_i$  in some  $\mathcal{O}(X_U)_{V_j}$ . Then we call  $\{f_i\}$  a *standard generating set* of  $\mathcal{O}_{\text{fin}}(X_U)$ . When  $U = T_B // G$  as before, we always assume that our standard generators are the restrictions of homogeneous elements of  $\mathcal{O}_{\text{alg}}(T_W)$ .

Let  $U \subset Q$ ,  $V$  and  $\{f_1, \dots, f_n\}$  be as above. We say that a  $G$ -equivariant homeomorphism  $\Psi : X_U \rightarrow X_U$  is *strong* if it lies over the identity of  $U$  and  $\Psi^* f_i = \sum_j a_{ij} f_j$  where the  $a_{ij}$  are in  $C^0(X_U)^G \simeq C^0(U)$ . We also require that the  $a_{ij}(q)$  induce a  $G$ -isomorphism of  $\mathcal{O}(X_q)_V$  for all  $q \in U$ . Then  $\Psi$  induces an algebraic isomorphism  $\Psi_q : X_q \rightarrow X_q$  for all  $q \in U$ . It is easy to see that the definition does not depend on our choice of  $V$  and the generators  $f_i$ . We call  $(a_{ij})$  a *matrix associated to  $\Psi$* . Using a partition of unity on  $U$  it is clear that  $\Psi$  is strong if and only if it is strong in a neighborhood of every  $q \in Q$ . In a neighborhood of any particular  $q$ , we may assume that the  $f_i$  restrict to a basis of  $\mathcal{O}(X_q)_V$ , in which case  $(a_{ij})$  is invertible in a neighborhood of  $q$ . Then  $\Phi^{-1}$  has matrix  $(a_{ij})^{-1}$  near  $q$ . Thus if  $\Phi$  is strong, so is  $\Phi^{-1}$ . Let  $\mathcal{A}_c(U)$  denote the group of strong  $G$ -homeomorphisms of  $X_U$  for  $U$  open in  $Q$ . Then  $\mathcal{A}_c$  is a sheaf of groups on  $Q$ .

We say that a vector field  $D$  on  $X_U$  is *formally holomorphic* if it annihilates the antiholomorphic functions on  $X_U$ . Let  $D$  be a continuous formally holomorphic vector field on  $X_U$ ,  $G$ -invariant, annihilating  $\mathcal{O}(X_U)^G$ . We say that  $D$  is *strongly continuous* (and write  $D \in \mathcal{LA}_c(U)$ ) if for any  $q \in U$  there is a neighborhood  $U'$  of  $q$  in  $U$  and a standard generating set  $f_1, \dots, f_n$  for  $\mathcal{O}_{\text{fin}}(X_{U'})$  such that  $D(f_i) = \sum d_{ij} f_j$  where the  $d_{ij}$  are in  $C^0(U')$ . We say that  $D$  *has matrix  $(d_{ij})$  over  $U'$* . The matrix is usually not unique. Clearly our definition of  $\mathcal{LA}_c(U)$  is independent of the choices made. We denote the corresponding sheaf by  $\mathcal{LA}_c$ .

*Remark 3.2.* Let  $D \in \mathcal{LA}_c(U)$  and  $q \in Q$ . Then  $D$  is tangent to  $F = X_q$  and acts algebraically on  $\mathcal{O}_{\text{alg}}(F)$ , hence lies in the space of  $G$ -invariant derivations  $\text{Der}_{\text{alg}}(F)^G$  of  $\mathcal{O}_{\text{alg}}(F)$ . Since  $\text{Der}_{\text{alg}}(F)^G$  is the Lie algebra of the algebraic group  $\text{Aut}(F)^G$ , the restriction of  $D$  to  $F$  can be integrated for all time. It follows that  $D$  is a complete vector field.

Let  $U$  be open in  $Q$ , let  $\epsilon > 0$  and let  $K$  be a compact subset of  $U$ . Let  $\mathbf{f} = \{f_1, \dots, f_n\}$  be a standard generating set of  $\mathcal{O}_{\text{fin}}(X_{U'})$  where  $U'$  is a neighborhood of  $K$ . Define

$$\Omega_{K, \epsilon, \mathbf{f}} = \{\Phi \in \mathcal{A}_c(U) : \|(a_{ij}) - I\|_K < \epsilon\}$$

where  $(a_{ij})$  is some matrix associated to  $\Phi$ . Here  $\|(a_{ij}) - I\|_K$  denotes the supremum of the matrix norm of  $(a_{ij}) - I$  over  $K$ . Let  $\mathbf{f}' = \{f'_1, \dots, f'_m\}$  be another standard generating set defined on a neighborhood of  $K$  in  $U$ .

**Lemma 3.3.** *Let  $\epsilon' > 0$ . Then there is an  $\epsilon > 0$  such that  $\Omega_{K, \epsilon, \mathbf{f}} \subset \Omega_{K, \epsilon', \mathbf{f}'}$ .*

*Proof.* We may assume that the  $f_i$  and  $f'_j$  are standard generating sets of  $\mathcal{O}_{\text{fin}}(U)$ . There are polynomials  $h_i$  with coefficients in  $\mathcal{O}(U)$  such that  $f'_i = h_i(f_1, \dots, f_n)$ ,  $1 \leq i \leq m$ . We may assume that  $\{f'_1, \dots, f'_s\}$  are the  $f'_i$  corresponding to an irreducible  $G$ -module  $V_t$ . Let  $\Phi \in \Omega_{K, \epsilon, \mathbf{f}}$  with corresponding matrix  $(a_{uv})$  such that  $\|(b_{uv})\|_K < \epsilon$  where  $(b_{uv}) = (a_{uv}) - I$ . Let  $r_i$  be the degree of  $h_i$ . Then for  $1 \leq i \leq s$  we have

$$(\Phi^* f'_i) - f'_i = h_i(\Phi^* f_1, \dots, \Phi^* f_n) - h_i(f_1, \dots, f_n) = \sum_{k,l=1}^n b_{kl} p_{kl} M_{kl}(f_1, \dots, f_n)$$

where the  $p_{kl}$  are polynomials in the  $a_{uv}$  of degree at most  $r_i - 1$  and the  $M_{k,l}$  are polynomials in the  $f_j$  with coefficients in  $\mathcal{O}(U)$  which are independent of the  $a_{uv}$  and  $b_{uv}$ . Since  $\Phi^* f'_i$  is a covariant corresponding to  $V_i$ , we can project the  $M_{kl}$  to  $\mathcal{O}(X_U)_{V_i}$  in which case we get  $\sum_{j=1}^s N_{jkl} f'_j$  where the  $N_{jkl}$  are in  $\mathcal{O}(U)$  and independent of the  $a_{uv}$  and  $b_{uv}$ . Hence

$$(\Phi^* f'_i) - f'_i = \sum_{j=1}^s \sum_{k,l=1}^n b_{kl} N_{jkl} p_{kl} f'_j.$$

Since the  $N_{jkl} p_{kl}$  are bounded on  $K$ , choosing  $\epsilon$  sufficiently small, we can force the terms  $\sum_{k,l=1}^n b_{kl} N_{jkl} p_{kl}$  to be close to 0. Hence there is an  $\epsilon > 0$  such that  $\Omega_{K,\epsilon,\mathbf{f}} \subset \Omega_{K,\epsilon',\mathbf{f}'}$ .  $\square$

By the lemma, we get the same neighborhoods of the identity in  $\mathcal{A}_c(U)$  from any standard generating set of  $\mathcal{O}_{\text{fin}}(X_{U'})$  where  $U'$  is a neighborhood of  $K$ . Thus we can talk about neighborhoods of the identity without specifying the  $\mathbf{f}$  in question. We then have a well-defined topology on  $\mathcal{A}_c(U)$  where  $\Phi$  is close to  $\Phi'$  if  $\Phi'\Phi^{-1}$  is close to the identity.

Let  $U$ ,  $K$  and the  $f_i$  be as above. Define

$$\Omega'_{K,\epsilon,\mathbf{f}} = \{D \in \mathcal{LA}_c(U) \mid D(f_i) = \sum d_{ij} f_j \text{ and } \|(d_{ij})\|_K < \epsilon\}$$

where  $D$  has (continuous) matrix  $(d_{ij})$  defined on a neighborhood of  $K$ . As before, the  $\Omega'_{K,\epsilon,\mathbf{f}}$  give a basis of neighborhoods of 0 and define a topology on  $\mathcal{LA}_c(U)$ , independent of the choice of  $\mathbf{f}$ .

**Proposition 3.4.** *Let  $U$  be open in  $Q$  and let  $\{f_1, \dots, f_n\}$  be a standard generating set for  $\mathcal{O}_{\text{fin}}(U)$ .*

- (1) *Let  $D$  be a  $G$ -invariant formally holomorphic vector field on  $X_U$  which annihilates  $\mathcal{O}(U)$  such that  $D(f_i) = \sum d_{ij} f_j$  for  $d_{ij} \in C^0(U)$ . Then  $D$  is continuous, i.e.,  $D \in \mathcal{LA}_c(U)$ .*
- (2)  *$\mathcal{LA}_c$  is a sheaf of Lie algebras and a module over the sheaf of germs of continuous functions on  $Q$ .*
- (3)  *$\exp: \mathcal{LA}_c(U) \rightarrow \mathcal{A}_c(U)$  is continuous.*
- (4)  *$\mathcal{LA}_c(U)$  is a Fréchet space.*

*Proof.* Let  $D$  be as in (1) and let  $x_0 \in X_U$ . There is a subset, say  $f_1, \dots, f_r$ , of the  $f_i$  and holomorphic invariant functions  $h_{r+1}, \dots, h_s$  such that the  $z_i = f_i - f_i(x_0)$  and  $z_j = h_j - h_j(x_0)$  are local holomorphic coordinates at  $x_0$ . Then, near  $x_0$ ,  $D$  has the form  $\sum a_i \partial / \partial z_i$  where each  $a_i = D(f_i)$  is continuous. Hence  $D$  is continuous giving (1). Let  $D, D' \in \mathcal{LA}_c(U)$  with matrices  $(d_{ij})$  and  $(d'_{ij})$ . Let  $(e_{ij})$  be their matrix bracket. Then  $[D, D']$  is  $G$ -invariant, annihilates  $\mathcal{O}(U)$  and sends  $f_i$  to  $\sum e_{ij} f_j$ . Hence we have (2). Part (3) is clear.

The topology on  $\mathcal{LA}_c(U)$ ,  $U$  open in  $Q$ , is defined by countably many seminorms, hence  $\mathcal{LA}_c(U)$  is a metric space and it is Fréchet if it is complete. Let  $D_k$  be a Cauchy sequence in  $\mathcal{LA}_c(U)$ . Let  $K \subset U$  be a compact neighborhood of  $q \in U$ . There are matrices  $(d_{ij}^k)$  of elements of  $C^0(K)$  such that  $D_k(f_i) = \sum d_{ij}^k f_j$  over  $K$ . Since  $\{D_k\}$  is Cauchy, we may assume that  $\|(d_{ij}^k) - (d_{ij}^l)\|_K < 1/m$  for  $k, l > N_m$ ,  $m \in \mathbb{N}$ . Then  $\lim_{k \rightarrow \infty} d_{ij}^k = d_{ij} \in C^0(K)$  for all  $i, j$ . It follows that the pointwise limit of the  $D_k$  exists and is a formally holomorphic vector field  $D$  annihilating the invariants such that  $D(f_i) = \sum d_{ij} f_j$ . By (1),  $D$  is of type  $\mathcal{LA}_c$  over the interior of  $K$ . It follows that  $\mathcal{LF}(U)$  is complete.  $\square$

#### 4. LOGARITHMS IN $\mathcal{A}_c$

Let  $U$  be an open subset of  $Q$  isomorphic to  $T_B = G \times^H B$  where  $H$  is a reductive subgroup of  $G$  and  $B$  is an  $H$ -saturated neighborhood of the origin in an  $H$ -module  $W$ . Let  $f_1, \dots, f_n$  be a standard generating set for  $\mathcal{O}_{\text{fin}}(X_U)^G$  consisting of the restrictions to  $X_U$  of homogeneous polynomials in  $\mathcal{O}_{\text{alg}}(T_W)$ . Consider polynomial relations of the  $f_i$  with coefficients in  $\mathcal{O}(U)$ .



These are generated by the relations with coefficients in  $\mathcal{O}_{\text{alg}}(T_W)^G$ . Let  $h_1, \dots, h_m$  be generating relations of this type. Let  $N$  be a bound for the degree of the  $h_j$ . Now take the covariants which correspond to all the irreducible  $G$ -representations occurring in the span of the monomials of degree at most  $N$  in the  $f_i$ . Let  $\{f_\alpha\}$  be a set of generators for these covariants and let  $K \subset U$  be compact. Let  $\Phi \in \mathcal{A}_c(U)$ . Then  $\Phi^* f_\alpha = \sum c_{\alpha,\beta} f_\beta$  where the  $c_{\alpha,\beta} \in C^0(U)$ . We also have that  $\Phi^* f_i = \sum a_{ij} f_j$  where the  $a_{ij} \in C^0(U)$ . We fix a neighborhood  $\Omega$  of the identity in  $\mathcal{A}_c(U)$  such that  $\Phi \in \Omega$  implies that  $\|(c_{\alpha,\beta}) - I\|_K < 1/3$  and that  $\|(a_{ij}) - I\|_K < 1/2$ . For  $\Phi \in \Omega$  let  $\Lambda'$  denote  $\text{Id} - \Phi^*$ . Then the formal power series  $S(\Lambda')$  for  $\log \Phi^*$  is  $-\Lambda' - (1/2)(\Lambda')^2 - 1/3(\Lambda')^3 - \dots$ .

Now we restrict to a fiber  $F = X_q$ ,  $q \in K$ . Let  $M$  denote the span of the covariants  $f_\alpha$  restricted to  $F$ . Then  $M$  is finite dimensional and we give it the usual euclidian topology. Let  $\Lambda$  denote the restriction of  $\Lambda'$  to  $M$ .

**Lemma 4.1.** *Let  $m \in M$ . Then the series  $S(\Lambda)(m)$  converges in  $M$ .*

*Proof.* We have  $m = \sum a_\alpha f_\alpha|_F$  where the  $a_\alpha \in \mathbb{C}$ . Then  $\Lambda(m) = \sum_{\alpha,\beta} (\delta_{\alpha,\beta} - c_{\alpha,\beta}(q)) a_\beta f_\beta|_F$ . Let  $C$  denote  $(c_{\alpha,\beta}(q))$ . Then  $\|I - C\| < 1/3$ . By induction,  $\Lambda^k$  acts on  $\sum_\alpha a_\alpha f_\alpha|_F$  via the matrix  $(I - C)^k$ , where  $\|(I - C)^k\| < (1/3)^k$ . Let

$$C' = - \sum_{k=1}^{\infty} (I - C)^k.$$

Then  $S(\Lambda)(m)$  converges to  $\sum_{\alpha,\beta} C'_{\alpha,\beta} a_\beta f_\beta|_F \in M$ . □

For  $f \in M$ , define  $D(f)$  to be the limit of  $S(\Lambda)(f)$ . Then  $D$  is a  $G$ -equivariant linear endomorphism of  $M$ .

**Proposition 4.2.** *Suppose that  $m_1$ ,  $m_2$  and  $m_1 m_2$  are in  $M$ . Then*

$$D(m_1 m_2) = D(m_1) m_2 + m_1 D(m_2).$$

*Proof.* By [Pra86, Proof of Theorem 4]

$$\Lambda^k(m_1 m_2) = \sum_{l=0}^{2k} \sum_{n=0}^{\ell} c_{k\ell n} \Lambda^n(m_1) \Lambda^{\ell-n}(m_2)$$

where  $c_{k\ell n}$  is the coefficient of  $x^n y^{\ell-n}$  in  $(x + y - xy)^k$ . We know that  $\Lambda^k$  is given by the action of the matrix  $(I - C)^k$  where  $\|I - C\| < 1/3$ . The series  $\sum 1/k (x + y - xy)^k$  converges absolutely when  $x$  and  $y$  have absolute value at most  $1/3$ . Thus we may make a change in the order of summation:

$$D(m_1 m_2) = \sum_{k=1}^{\infty} \sum_{l=0}^{2k} \sum_{n=0}^{\ell} \frac{1}{k} c_{k\ell n} \Lambda^n(m_1) \Lambda^{\ell-n}(m_2) = \sum_{\ell=0}^{\infty} \sum_{n=0}^{\ell} \sum_{k=1}^{\infty} \frac{1}{k} c_{k\ell n} \Lambda^n(m_1) \Lambda^{\ell-n}(m_2).$$

By [Pra86, Proof of Theorem 4] the (actually finite) sum  $\sum_{k=1}^{\infty} (1/k) c_{k\ell n}$  equals 0 unless we have  $\ell > 0$  and  $(n = 0 \text{ or } n = \ell)$ , in which case the value is  $1/\ell$ . Hence

$$D(m_1 m_2) = - \sum_{\ell=1}^{\infty} \frac{1}{\ell} (\Lambda^\ell(m_1) m_2 + m_1 \Lambda^\ell(m_2)) = D(m_1) m_2 + m_1 D(m_2).$$

□

**Proposition 4.3.** *Let  $D: M \rightarrow M$  be as above. Then  $D$  extends to a  $G$ -equivariant derivation of  $\mathcal{O}_{\text{alg}}(F)$ .*

*Proof.* Let  $R = \mathbb{C}[z_1, \dots, z_n]$ . We have a surjective morphism  $\rho: R \rightarrow \mathcal{O}_{\text{alg}}(F)$  sending  $z_i$  to  $f_i|_F$ ,  $i = 1, \dots, n$ . The kernel  $J$  of  $\rho$  is generated by polynomials of degree at most  $N$  (the  $h_j$  considered as elements of  $R$ ). Let  $E$  denote the derivation of  $R$  which sends  $z_i$  to  $\sum d'_{ij} z_j$  where  $(d'_{ij})$  is the logarithm of  $(a_{ij}(q))$ . Recall that  $\|(a_{ij}) - I\|_K < 1/2$ . By construction,  $E$  on the span of the  $z_i$  is the pull-back of  $D$  on the span of the  $f_i|_F$ . By Proposition 4.2,  $E$  restricted to polynomials of degree at most  $N$  is the pull-back of  $D$  restricted to polynomials of degree at most  $N$  in the  $f_i|_F$ . Hence  $E$  preserves the span of the elements of degree at most  $N$  in  $J$ . Since these elements generate  $J$ , we see that  $E$  preserves  $J$ . Hence  $E$  induces a derivation of  $R/J$ , i.e.,  $D$  extends to a  $G$ -invariant derivation of  $\mathcal{O}_{\text{alg}}(F)$ .  $\square$

**Corollary 4.4.** *Let  $\Phi \in \Omega$  and let  $U'$  denote the interior of our compact subset  $K \subset U$ . There is a  $D \in \mathcal{LA}_c(U')$  such that  $\exp(D) = \Phi|_{X_{U'}}$ . The mapping  $\Omega \ni \Phi \rightarrow D \in \mathcal{LA}_c(U')$  is continuous.*

*Proof.* For  $q \in U'$ , let  $D_q$  be the  $G$ -equivariant derivation of  $\mathcal{O}_{\text{alg}}(X_q)$  constructed above. Let  $D$  be the vector field on  $X_{U'}$  whose value on  $X_q$  is  $D_q$ ,  $q \in U'$ . Then  $D(f_i) = \sum d_{ij} f_j$  where  $(d_{ij}) = \log(a_{ij})$ . By Proposition 3.4,  $D \in \mathcal{LA}_c(U')$ . By construction,  $\exp(D_q) = \Phi_q$  for all  $q \in U'$ . Hence  $\exp D = \Phi|_{X_{U'}}$ . The continuity of  $\Phi \mapsto D$  is clear since  $(d_{ij}) = \log(a_{ij})$ .  $\square$

**Definition 4.5.** Let  $U \subset Q$  be open and let  $f_1, \dots, f_n$  be a standard generating set of  $\mathcal{O}_{\text{fin}}(X_U)$ . Let  $U' \subset U$  be open with  $\overline{U'} \subset U$ . We say that  $\Phi \in \mathcal{A}_c(U)$  admits a logarithm in  $\mathcal{LA}_c(U')$  if the following hold.

- (1)  $\Phi^* f_i = \sum a_{ij} f_j$  where  $\|(a_{ij}) - I\|_{\overline{U'}} < 1/2$ .
- (2) There is a  $D \in \mathcal{LA}_c(U')$  such that  $D(f_i) = \sum d_{ij} f_j$  on  $X_{U'}$  where  $(d_{ij}) = \log(a_{ij})$ .

Note that  $(a_{ij})$  is not unique. The condition is that some  $(a_{ij})$  corresponding to  $\Phi$  satisfies (1) and (2).

*Remarks 4.6.* The formal series  $\log \Phi^*$ , when applied to any  $f_i$ , converges to  $D(f_i)$ . Hence  $D$  is independent of the choice of  $(a_{ij})$ . Properties (1) and (2) imply that  $\exp D = \Phi$  over  $U'$ . Note that  $\|(d_{ij})\|_{\overline{U'}} < \log 2$  and  $(d_{ij})$  is the unique matrix satisfying this property whose exponential is  $(a_{ij})$ .

Corollary 4.4 and its proof imply the following result.

**Theorem 4.7.** *Let  $K \subset U \subset Q$  where  $K$  is compact and  $U$  is open. Then there is a neighborhood  $\Omega$  of the identity in  $\mathcal{A}_c(U)$  and a neighborhood  $U'$  of  $K$  in  $U$  such that every  $\Phi \in \Omega$  admits a logarithm  $D = \log \Phi$  in  $\mathcal{LA}_c(U')$ . The mapping  $\Phi \rightarrow \log \Phi$  is continuous.*

**Corollary 4.8.** *Let  $\Phi_n$  be a Cauchy sequence in  $\mathcal{A}_c(U)$ . Then  $\Phi_n \rightarrow \Phi \in \mathcal{A}_c(U)$ .*

*Proof.* Since this is a local question, we can assume that we have a standard generating set  $\{f_i\}$  for  $\mathcal{O}_{\text{fin}}(U)$ . Let  $q \in U$  and let  $U'$  be a relatively compact neighborhood of  $q$  in  $U$ . Then there is a neighborhood  $\Omega$  of the identity in  $\mathcal{A}_c(U)$  such that any  $\Psi \in \Omega$  admits a logarithm in  $\mathcal{LA}_c(U')$ . Let  $\Omega_0$  be a smaller neighborhood of the identity with  $\overline{\Omega_0} \subset \Omega$ . There is an  $N \in \mathbb{N}$  such that  $n \geq N$  implies that  $\Phi_N^{-1} \Phi_n \in \Omega_0$ , hence  $\log(\Phi_N^{-1} \Phi_n) = D_n \in \mathcal{LA}_c(U')$ , and  $D_n$  converges to an element  $D \in \mathcal{LA}_c(U')$  by Proposition 3.4. Set  $\Phi = \Phi_N \exp D \in \mathcal{A}(U')$ . Since  $\exp D_n = \Phi_N^{-1} \Phi_n$  over  $U'$  we have  $\Phi_n \rightarrow \Phi$  in  $\mathcal{A}_c(U')$ .  $\square$

## 5. HOMOTOPIES IN $H^1(Q, \mathcal{A}_c)$

We establish our main technical result concerning homotopies in  $H^1(Q, \mathcal{A}_c)$ . We give proofs of Theorems 1.1 and 1.6 and a special case of Theorem 1.5.

Let  $\Phi(t) \in \mathcal{A}_c(U)$ ,  $t \in C$ , where  $C$  is a topological space. We say that  $\Phi(t)$  is continuous if relative to a standard generating set,  $\Phi(t)$  has corresponding matrices  $(a_{ij}(t, q))$  where each  $a_{ij}$  is continuous in  $t$  and  $q \in U$ . (It is probably false that every continuous map  $C \rightarrow \mathcal{A}_c(U)$

is continuous in our sense.) Let  $\mathfrak{A}_c(U)$  denote the set of all continuous paths  $\Phi(t) \in \mathcal{A}_c(U)$ ,  $t \in [0, 1]$ , starting at the identity. We have a topology on  $\mathfrak{A}_c(U)$  as in §3 and  $\mathfrak{A}_c(U)$  is a topological group. When we talk of homotopies in  $\mathfrak{A}_c(U)$  we mean that the corresponding families with parameter space  $[0, 1]^2$  are continuous as above. We define continuous families of elements of  $\mathcal{LA}_c(U)$  similarly. One defines  $\mathfrak{A}(U)$  similarly to  $\mathfrak{A}_c(U)$  where, of course, the relevant  $a_{ij}(t, q)$  are required to be holomorphic in  $q$  and continuous in  $t$ .

Here is our main technical result about  $\mathfrak{A}_c$ .

**Theorem 5.1.** (1) *The topological group  $\mathfrak{A}_c(Q)$  is pathwise connected.*  
 (2) *If  $U \subset Q$  is open, then  $\mathfrak{A}_c(Q)$  is dense in  $\mathfrak{A}_c(U)$ .*  
 (3)  *$H^1(Q, \mathfrak{A}_c) = 0$ .*

*Proof.* Let  $\Phi(t)$  be an element of  $\mathfrak{A}_c(Q)$ . Since  $\{0\}$  is a deformation retract of  $[0, 1]$ , there is a homotopy  $\Phi(s, t)$  with  $\Phi(0, t) = \Phi(t)$  and  $\Phi(1, t)$  the identity automorphism. Hence we have (1). For (2), let  $\Phi \in \mathfrak{A}_c(U)$ . Let  $K$  be a compact subset of  $U$  and  $U'$  a relatively compact neighborhood of  $K$  in  $U$ . It follows from Theorem 4.7 that there are  $0 = t_0 < t_1 < \dots < t_m = 1$  and continuous families  $D_j(s)$  in  $\mathcal{LA}_c(U')$  for  $s \in [0, t_{j+1} - t_j]$  such that, over  $U'$ ,  $\Phi(s + t_j) = \Phi(t_j) \exp(D_j(s))$ ,  $s \in [0, t_{j+1} - t_j]$ ,  $j = 0, \dots, m-1$ . Multiplying by a cutoff function, we can assume that the  $D_j(s)$  are in  $\mathcal{LA}_c(Q)$ . Then our formula gives an element of  $\mathfrak{A}_c(Q)$  which restricts to  $\Phi$  on a neighborhood of  $K$  and we have (2).

Let  $K \subset Q$  be compact which is of the form  $K' \cup K''$  where  $K'$  and  $K''$  are compact. Let  $U = U' \cup U''$  be a neighborhood of  $K$  where  $K' \subset U'$ ,  $K'' \subset U''$ . Let  $\Phi(t)$  be in  $Z^1(U, \mathfrak{A}_c)$  for the open covering  $\{U', U''\}$  of  $U$ . Then  $\Phi(t)$  is just an element in  $\mathfrak{A}_c(U' \cap U'')$ . By (2) we can write  $\Phi = \Psi_1 \Psi_2^{-1}$  where  $\Psi_1$  is defined over  $Q$  (hence over  $U'$ ) and  $\Psi_2$  is close to the identity over  $K' \cap K''$ . Then  $\Psi_2(t) = \exp D(t)$  where  $D(t) \in \mathcal{LA}_c(U' \cap U'')$  is a continuous family and  $D(0) = 0$ . Using a cutoff function again, we can find  $D_0(t) \in \mathcal{LA}_c(Q)$  which equals  $D(t)$  in a neighborhood of  $K' \cap K''$  and vanishes when  $t = 0$ . We have  $\Phi = \Psi_1 \Psi_2^{-1}$  where  $\Psi_2^{-1}$  is the exponential of  $D_0(t)$ . Thus the cohomology class of  $\Phi$  becomes trivial if we replace  $U'$  and  $U''$  by slightly smaller neighborhoods of  $K'$  and  $K''$ . Let  $H^1(K, \mathfrak{A}_c)$  denote the direct limit of  $H^1(U, \mathfrak{A}_c)$  for  $U$  a neighborhood of  $K$ . As in [Car58, §5], our result above shows that there is a sequence of compact sets  $K_1 \subset V_2 \subset K_2 \dots$  with  $V_n$  the interior of  $K_n$ ,  $Q = \bigcup K_n$  and  $H^1(K_n, \mathfrak{A}_c) = 0$  for all  $n$ .

Let  $\{U_i\}$  be an open cover of  $Q$  and  $\Phi_{ij} \in \mathfrak{A}_c(U_i \cap U_j)$  a cocycle. There are  $c_i^n \in \mathfrak{A}_c(U_i \cap V_n)$  such that  $\Phi_{ij} = (c_i^n)^{-1} c_j^n$  on  $U_i \cap U_j \cap V_n$ . Thus  $c_i^{n+1} (c_i^n)^{-1} = c_j^{n+1} (c_j^n)^{-1}$  on  $U_i \cap U_j \cap V_n$ . The  $c_i^{n+1} (c_i^n)^{-1}$  define a section  $d \in \mathfrak{A}_c(V_n)$ . By (2) there is a section  $d'$  of  $\mathfrak{A}_c(Q)$  which is arbitrarily close to  $d$  on  $K_{n-1}$ . Replace each  $c_i^{n+1}$  by  $(d')^{-1} c_i^{n+1}$ . Then  $c_i^{n+1}$  is very close to  $c_i^n$  on  $K_{n-1}$  and we can arrange that the limit as  $n \rightarrow \infty$  of the  $c_i^n$  converges on every compact subset to  $c_i \in \mathfrak{A}_c(U_i)$  such that  $\Phi_{ij} = c_i^{-1} c_j$ . We have used Corollary 4.8. This completes the proof of (3).  $\square$

Note that (3) says that for any homotopy of a cocycle  $\Phi_{ij}(t)$  starting at the identity there are  $c_i(t) \in \mathcal{A}_c(U_i)$  such that  $\Phi_{ij}(t) = c_i(t)^{-1} c_j(t)$  for all  $t \in [0, 1]$ . Hence  $\Phi_{ij}(t)$  is the trivial element in  $H^1(Q, \mathcal{A}_c)$  for all  $t$ . We now use a trick to show a similar result if we only assume that  $\Phi_{ij}(0) \in Z^1(Q, \mathcal{A})$ .

Let  $\Psi_{ij} \in Z^1(Q, \mathcal{A})$  for some open cover  $\{U_i\}$  of  $Q$ . By [KLS, Theorem 5.11], there is a Stein  $G$ -manifold  $Y$  with quotient  $Q$  corresponding to the  $\Psi_{ij}$ . Let  $X_i = X_{U_i}$  and  $Y_i = Y_{U_i}$ . Then there are  $G$ -biholomorphisms  $\Psi_i: X_i \rightarrow Y_i$  over the identity of  $U_i$  such that  $\Psi_i^{-1} \Psi_j = \Psi_{ij}$ .

Here is an analogue of the twist construction in Galois cohomology. We leave the proof to the reader.



**Lemma 5.2.** *Let  $\Psi_{ij} \in Z^1(Q, \mathcal{A})$  and  $\Phi_{ij} \in Z^1(Q, \mathcal{A}_c)$  be cocycles for the open cover  $\{U_i\}$  of  $Q$ . Let  $Y$  and  $\Psi_i: X_i \rightarrow Y_i$  be as above. The mapping  $\Phi_{ij} \mapsto \Psi_i \Phi_{ij} \Psi_j^{-1}$  induces an isomorphism of  $H^1(Q, \mathcal{A}_c)$  and  $H^1(Q, \mathcal{A}_c^Y)$  which sends the class  $\Psi_{ij}$  to the trivial class of  $H^1(Q, \mathcal{A}_c^Y)$ .*

**Corollary 5.3.** *Let  $\Phi_{ij}(t)$  be a homotopy of cocycles with values in  $\mathcal{A}_c(U_i \cap U_j)$  where  $\{U_i\}$  is an open cover of  $Q$ . Suppose that  $\Phi_{ij}(0)$  is holomorphic. Then there are  $c_i \in \mathfrak{A}_c(U_i)$  such that  $\Phi_{ij}(t) = c_i(t)^{-1} \Phi_{ij}(0) c_j(t)$  for all  $t$ .*

*Proof.* By Lemma 5.2 we may reduce to the case that  $\Phi_{ij}(0)$  is the identity, so we can apply Theorem 5.1.  $\square$

Let  $X, Y$  and the  $\Psi_i$  be as above. We say that a  $G$ -homeomorphism  $\Phi: X \rightarrow Y$  is *strong* if  $\Psi_i^{-1} \circ \Phi: X_i \rightarrow X_i$  is strong for all  $i$ , i.e., in  $\mathcal{A}_c(U_i)$ . It is easy to see that this does not depend upon the particular choice of the  $\Psi_i$ . Similarly one can define what it means for a family  $\Phi(t)$  of strong  $G$ -homeomorphisms to be continuous,  $t \in [0, 1]$ . Then we have the following nice result [KLS, Theorem 1.4].

**Theorem 5.4.** *Let  $\Phi: X \rightarrow Y$  be strongly continuous. Then there is a continuous family  $\Phi(t)$  of strong  $G$ -homeomorphisms from  $X$  to  $Y$  with  $\Phi(0) = \Phi$  and  $\Phi(1)$  holomorphic.*

*Proof of Theorem 1.1.* We have  $\Phi_{ij}, \Psi_{ij} \in Z^1(Q, \mathcal{A})$  and  $c_i \in \mathcal{A}_c(U_i)$  satisfying  $\Phi_{ij} = c_i \Psi_{ij} c_j^{-1}$ . Using Lemma 5.2 we may assume that  $\Phi_{ij}$  is the trivial class. Then the  $c_i$  are the same thing as a strong  $G$ -homeomorphism  $\Theta: X \rightarrow Y$  where  $Y$  is the Stein  $G$ -manifold corresponding to the  $\Psi_{ij}$  (after our twisting). By Theorem 5.4 not only are there  $d_i \in \mathcal{A}(U_i)$  such that  $\Psi_{ij} = d_i d_j^{-1}$ , but the  $d_i$  are  $e_i(1)$  where  $e_i(t)$  is a path in  $\mathcal{A}_c(U_i)$  starting at  $c_i$  and ending at  $d_i$ . The  $d_i$  correspond to a  $G$ -biholomorphism of  $X$  and  $Y$  over  $Q$ .  $\square$

We now prepare to prove Theorem 1.6.

**Lemma 5.5.** *Let  $\Phi \in \mathfrak{A}_c(Q)$  such that  $\Phi(1)$  is holomorphic. Then  $\Phi$  is homotopic to  $\Phi' \in \mathfrak{A}(Q)$  where  $\Phi'(1) = \Phi(1)$ .*

*Proof.* We have to make use of a sheaf of groups  $\mathcal{F}$  on  $Q$  which is a subsheaf of the sheaf of  $G$ -diffeomorphisms of  $X$  which induce the identity on  $Q$  and are algebraic isomorphisms on the fibers of  $p$ . See [KLS, Ch. 6]. We give  $\mathcal{F}(U)$  the usual  $C^\infty$ -topology. Let  $\mathfrak{F}(U)$  denote the sheaf of homotopies  $\Psi(t)$  of elements of  $\mathcal{F}(U)$ ,  $t \in [0, 1]$ , where  $\Psi(0)$  is the identity and  $\Psi(1)$  is holomorphic. Then [KLS, Theorem 10.1] tells us that  $\mathfrak{F}(Q)$  is pathwise connected. Hence for  $\Phi \in \mathfrak{F}(Q)$  there is a homotopy  $\Psi(s) \in \mathfrak{F}(Q)$  such that  $\Psi(0) = \Phi$  and  $\Psi(1)$  is the identity. Then  $\Psi(s)$  evaluated at  $t = 1$  is a homotopy from  $\Phi(1)$  to the identity in  $\mathcal{A}(Q)$ , establishing the lemma when  $\Phi \in \mathfrak{F}(Q)$ .

We now use a standard trick. Let  $\Delta$  denote a disk in  $\mathbb{C}$  containing  $[0, 1]$  with trivial  $G$ -action. Then  $\Delta \times X$  has quotient  $\Delta \times Q$  with the obvious quotient mapping. Let  $\rho: \Delta \rightarrow [0, 1]$  be continuous such that  $\rho$  sends a neighborhood of 0 to 0 and a neighborhood of 1 to 1. For  $(z, x) \in \Delta \times X$ , define  $\Psi(z, x) = (z, \Phi(\rho(z), x))$ . Then  $\Psi \in \tilde{\mathcal{A}}_c(\Delta \times Q)$  where  $\tilde{\mathcal{A}}_c = \mathcal{A}_c^{\Delta \times X}$ . Moreover,  $\Psi$  is the identity on the inverse image of a neighborhood of  $\{0\} \times Q$  and is holomorphic on the inverse image of a neighborhood of  $\{1\} \times Q$ . By [KLS, Theorem 8.7] we can find a homotopy  $\Psi(s)$  which starts at  $\Psi$  and ends up in  $\mathcal{F}(\Delta \times Q)$ . Moreover, the proof shows that we can assume that the elements of the homotopy are unchanged over a neighborhood of  $\{0, 1\} \times Q$ . Restricting  $\Psi(1)$  to  $[0, 1] \subset \Delta$  we have an element in  $\mathfrak{F}(Q)$  which at time 1 is still  $\Phi(1)$ . Then we can apply the argument above.  $\square$

*Proof of Theorem 1.6.* By Lemma 5.2 we may assume that  $\Phi_{ij}(0)$  is the identity cocycle. Since  $H^1(Q, \mathfrak{A}_c)$  is trivial, there are  $c_i \in \mathfrak{A}_c(U_i)$  such that  $\Phi_{ij}(t) = c_i(t) c_j(t)^{-1}$  for  $t \in [0, 1]$ . Now the  $c_i(1)$  define a strongly continuous  $G$ -homeomorphism from  $X$  to the Stein  $G$ -manifold  $Y$

corresponding to  $\Phi_{ij}(1)$ . By Theorem 5.4 there is a homotopy  $c_i(t)$ ,  $1 \leq t \leq 2$ , such that the  $c_i(2)$  are holomorphic and split  $\Phi_{ij}(1)$ . Reparameterizing, we may reduce to the case that the original  $c_i(t)$  are holomorphic for  $t = 1$ . Now apply Lemma 5.5 to  $\Psi_{ij}(t) = c_i(t)c_j(t)^{-1}$ .  $\square$

## 6. $H^1(Q, \mathcal{A}) \rightarrow H^1(Q, \mathcal{A}_c)$ IS A BIJECTION

We give a proof Theorem 1.2. We are given an open cover  $\{U_i\}$  of  $Q$  and  $\Phi_{ij} \in Z^1(Q, \mathcal{A}_c)$ . We want to find  $c_i \in \mathcal{A}_c(U_i)$  such that  $c_i^{-1}\Phi_{ij}c_j$  is holomorphic. We may assume that the  $U_i$  are relatively compact, locally finite and Runge. We say that an open set  $U \subset Q$  is *good* if there are sections  $c_i \in \mathcal{A}_c(U_i \cap U)$  such that  $c_i^{-1}\Phi_{ij}c_j$  is holomorphic on  $U_{ij} \cap U$  for all  $i$  and  $j$  where  $U_{ij}$  denotes  $U_i \cap U_j$ . This says that  $\{\Phi_{ij}\}$  is cohomologous to a holomorphic cocycle on  $U$ . The goal is to show that  $Q$  is good. It is obvious that small open subsets of  $Q$  are good.

**Lemma 6.1.** *Suppose that  $Q = Q' \cup Q''$  where  $Q'$  and  $Q''$  are good and  $Q' \cap Q''$  is Runge in  $Q$ . Then  $Q$  is good.*

*Proof.* By hypothesis, we have  $c'_i \in \mathcal{A}_c(Q' \cap U_i)$  and  $c''_i \in \mathcal{A}_c(Q'' \cap U_i)$  such that

$$\Psi'_{ij} = (c'_i)^{-1}\Phi_{ij}c'_j, \text{ and } \Psi''_{ij} = (c''_i)^{-1}\Phi_{ij}c''_j \text{ are holomorphic.}$$

Then on  $U_{ij} \cap Q' \cap Q''$  we have

$$\Psi''_{ij} = h_i^{-1}\Psi'_{ij}h_j \text{ where } h_i = (c'_i)^{-1}c''_i.$$

The  $\Psi'_{ij}$  are a holomorphic cocycle for the covering  $U_i \cap Q'$  of  $Q'$ , hence they correspond to a Stein  $G$ -manifold  $X'$  with quotient  $Q'$ . Similarly the  $\Psi''_{ij}$  give us  $X''$ , and  $X'$  and  $X''$  are locally  $G$ -biholomorphic to  $X$  over  $Q'$  and  $Q''$ , respectively. The  $h_i$  give us a strong  $G$ -homeomorphism  $h: X' \rightarrow X''$ , everything being taken over  $Q' \cap Q''$ . By Theorem 5.4 there is a homotopy  $h(t, x)$  with  $h(0, x) = h(x)$  and  $h(1, x)$  holomorphic. Let  $k(x)$  denote  $h(1, x)$ . Then  $k$  corresponds to a family  $k_i$  homotopic to the family  $h_i$ .

Now just consider the space  $U_i$  covered by the two open sets  $U_i \cap Q'$  and  $U_i \cap Q''$ . Then  $h_i$  and  $k_i$  are defined on the intersection of the two open sets and are homotopic where  $h_i$  is cohomologous to the trivial cocycle since  $h_i = (c'_i)^{-1}c''_i$ . By Corollary 5.3 and Theorem 1.1 the cohomology class represented by  $k_i(x)$  is holomorphically trivial. Hence there are holomorphic sections  $h'_i$  and  $h''_i$  such that  $k_i = (h'_i)^{-1}h''_i$  on  $U_i \cap Q' \cap Q''$ . Then  $h'_i\Psi'_{ij}(h'_j)^{-1} = h''_i\Psi''_{ij}(h''_j)^{-1}$  on  $U_{ij} \cap Q' \cap Q''$ . We construct a holomorphic cocycle  $\Psi_{ij}$  on  $U_{ij}$  by  $\Psi_{ij} = h'_i\Psi'_{ij}(h'_j)^{-1}$  on  $U_{ij} \cap Q'$  and  $h''_i\Psi''_{ij}(h''_j)^{-1}$  on  $U_{ij} \cap Q''$ .

Using Lemma 5.2 we may reduce to the case that  $\Psi_{ij}$  is the trivial cocycle. As in the beginning of the proof there are  $c'_i \in \mathcal{A}_c(Q' \cap U_i)$  and  $c''_i \in \mathcal{A}_c(Q'' \cap U_i)$  such that

$$\Phi_{ij}|_{X'} = c'_i(c'_j)^{-1} \text{ and } \Phi_{ij}|_{X''} = c''_i(c''_j)^{-1}$$

where  $X' = X_{Q'}$  and  $X'' = X_{Q''}$ . Let  $h_i = (c'_i)^{-1}c''_i$ . Then  $h_i = h_j$  on  $U_{ij} \cap Q' \cap Q''$ , hence we have a section  $h \in \mathcal{A}_c(Q' \cap Q'')$ , and this section gives the same cohomology class as  $\Phi_{ij}$  (use the open cover  $\{Q' \cap U_i, Q'' \cap U_i\}$ ). By Theorem 5.4,  $h$  is homotopic to an element  $\tilde{h} \in \mathcal{A}(Q' \cap Q'')$ , and this holomorphic section gives the same cohomology class by Corollary 5.3. Since going to a refinement of an open cover is injective on  $H^1$ , we see that our original  $\Phi_{ij}$  differs from a holomorphic cocycle by a coboundary. Thus  $Q$  is good.  $\square$

*Proof of Theorem 1.2.* Using Lemma 6.1 as in [Car58, §5] we can show that there is a cover of  $Q$  by compact subsets  $K_n$  such that  $K_1 \subset V_2 \subset K_2 \dots$  where  $V_j$  is the interior of  $K_j$  and such that a neighborhood of every  $K_n$  is good. We can assume that  $U_i \cap V_n \neq \emptyset$  implies that  $U_i \subset V_{n+1}$ . This is possible by replacing  $\{K_n\}$  by a subsequence. For each  $n$  we choose  $c_i^n \in \mathcal{A}_c(U_i \cap V_n)$  such that

$$(c_i^n)^{-1}\Phi_{ij}c_j^n = \Psi_{ij}^n \text{ is holomorphic on } U_{ij} \cap V_n.$$

Then

$$\Psi_{ij}^n = (d_i^n)^{-1} \Psi_{ij}^{n+1} d_j^n \text{ on } U_{ij} \cap V_n$$

where  $d_i^n = (c_i^{n+1})^{-1} c_i^n$  gives a strongly continuous map from the Stein  $G$ -manifold  $Y_n$  over  $V_n$  obtained using the  $\Psi_{ij}^n$  to the Stein  $G$ -manifold  $Y_{n+1}$  obtained using the  $\Psi_{ij}^{n+1}$ . We know that the map is homotopic to a holomorphic one. Hence there are homotopies  $d_i^n(t)$  on  $U_i \cap V_n$  such that

- (1)  $\Psi_{ij}^n = (d_i^n(t))^{-1} \Psi_{ij}^{n+1} d_j^n(t)$  on  $U_i \cap U_j \cap V_n$ , for all  $t$ .
- (2)  $d_i^n(0) = (c_i^{n+1})^{-1} c_i^n$ .
- (3) The  $d_i^n(1)$  give a  $G$ -equivariant biholomorphic map from  $Y_n$  to  $Y_{n+1}$  over  $\text{Id}_{V_n}$ .

Without changing the  $c_i^n$  we may replace the  $c_i^{n+1}$  by sections  $\tilde{c}_i^{n+1}$  such that

- (4)  $\tilde{\Psi}_{ij}^{n+1} = (\tilde{c}_i^{n+1})^{-1} \Phi_{ij} \tilde{c}_j^{n+1}$  is holomorphic on  $U_i \cap U_j \cap V_{n+1}$ .
- (5)  $\tilde{c}_i^{n+1} = c_i^n$  on  $U_i \cap V_{n-2}$ .

It suffices to set  $\tilde{c}_i^{n+1} = c_i^{n+1}$  if  $U_i \cap V_{n-1} = \emptyset$  and if not, then  $U_i \subset V_n$ , and we can set

$$\tilde{c}_i^{n+1} = c_i^{n+1} \cdot d_i^n(\lambda(x)),$$

where  $\lambda: V_n \rightarrow [0, 1]$  is continuous, 0 for  $x \in V_{n-2}$  and 1 for  $x \notin V_{n-1}$ . Then one has (4) and (5). Thus we can arrange that  $c_i^{n+1} = c_i^n$  in  $U_i \cap V_{n-2}$ , hence we obviously have convergence of the  $c_i^n$  to a continuous section  $c_i$  such that  $(c_i)^{-1} \Phi_{ij} c_j$  is holomorphic.  $\square$

We now have Theorems 1.1 and 1.2 which imply Corollary 1.3, i.e., that  $H^1(Q, \mathcal{A}) \rightarrow H^1(Q, \mathcal{A}_c)$  is a bijection.

*Proof of Theorem 1.5.* This is immediate from Corollary 5.3 and Theorem 1.2  $\square$

We end with the analogue of an approximation theorem of Grauert.

**Theorem 6.2.** *Let  $U \subset Q$  be Runge. Suppose that  $\Phi: X_U \rightarrow Y_U$  is biholomorphic and  $G$ -equivariant inducing  $\text{Id}_U$ . Here  $X$  and  $Y$  are locally  $G$ -biholomorphic over  $Q$ . Then  $\Phi$  can be arbitrarily closely approximated by  $G$ -biholomorphisms of  $X$  and  $Y$  over  $\text{Id}_Q$  if and only if this is true for strong  $G$ -homeomorphisms of  $X$  and  $Y$ .*

*Proof.* Let  $K \subset U$  be compact and let  $\Phi \in \text{Mor}(X_U, Y_U)^G$  be our holomorphic  $G$ -equivariant map inducing  $\text{Id}_U$ . We can find a relatively compact open subset  $U'$  of  $U$  which contains  $K$  and is Runge in  $Q$ . By hypothesis, there is a strong  $G$ -homeomorphism  $\Psi: X \rightarrow Y$  which is arbitrarily close to  $\Phi$  over  $\overline{U'}$ . Then  $\Psi^{-1}\Phi = \exp D$  where  $D \in \mathcal{L}\mathcal{A}_c(U')$ , hence  $\Psi'$  and  $\Phi'$  are homotopic, where  $\Phi'$  is the restriction of  $\Phi$  to  $U'$  and similarly for  $\Psi'$ . Now  $\Psi$  is homotopic to a biholomorphic  $G$ -equivariant map  $\Theta: X \rightarrow Y$  inducing  $\text{Id}_Q$ , and  $\Psi'$  is homotopic to the restriction  $\Theta'$  of  $\Theta$  to  $U'$ . Then  $(\Phi')^{-1}\Theta'$  is holomorphic and homotopic to the identity section over  $U'$ . Since the end points of the homotopy are holomorphic, by Theorem 1.6 we can find a homotopy all of whose elements are holomorphic. By [KLS, Theorem 10.1] there is a section  $\Delta \in \mathcal{A}(Q)$  which is arbitrarily close to  $(\Phi')^{-1}\Theta'$  on  $U'$ . Then  $\Theta\Delta^{-1}$ , restricted to  $U'$ , is arbitrarily close to  $\Phi'$ , hence this is true over  $K$ . This establishes the theorem.  $\square$

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